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Manifolds and Group actions

Homework 3

Mandatory Exercise 1. (8 Points)

a) Show that if $f : M \to \mathbb{R}$ is a map that is constant in a neighborhood of $p \in M$, that f is smooth in that neighborhood and that $v(f) = 0$ for all $v \in T_pM$.

b) Let $f, g : M \to \mathbb{R}$ be smooth maps and $p \in M$ such that $f(p) = g(p) = 0$. Show that $v(fg) = 0$ for all $v \in T_pM$.

Mandatory Exercise 2. (4 Points)
Let $\varphi = (\varphi_1, \ldots, \varphi_n) : W \to \mathbb{R}^n$ be a chart of M and $f : M \to \mathbb{R}$ a smooth function. Show that for $p \in W$

$$(df)_p = \sum_{i=1}^{n} \frac{\partial \hat{f}}{\partial \varphi_i}(\varphi(p))(d\varphi_i)_p,$$

where $\hat{f} = f \circ \varphi^{-1}$ is the function in the local coordinates induced by φ.

Mandatory Exercise 3. (8 Points)

a) Consider the antipodal map $f : S^2 \to S^2$ by $f(x) = -x$. Let N be the north pole and S the south pole. Compute the derivative $(df)_N : T_N S^2 \to T_S S^2$.

For the rest of this exercise we view $S^2 \subset \mathbb{C} \times \mathbb{R}$. Let $n \in \mathbb{N}$ and let $g_n : S^2 \to S^2$ be defined by

$$g_n(z, t) = \begin{cases} (\frac{z^n}{|z|^{n-1}}, t) & \text{if} \quad z \neq 0 \\ (0, t) & \quad\quad z = 0 \end{cases}$$

b) Show that g_n is a smooth map.

c) A point $(z, t) \in S^2$ is a fixed point of g_n if $g_n(p) = p$. Compute the set of fixed points of g_n.

d) For each fixed point p of g_n, the differential is a map $(dg_n)_p : T_pS^2 \to T_pS^2$. Compute this map explicitely for all fixed points.

Suggested Exercise 1. (0 Points)
Let M be a smooth manifold and $\phi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^n$ two charts. Let V be a vector field on M. Then in local coordinates we can write that

$$V(p) = \sum_{i=1}^{n} V_i^{\phi}(p) \frac{\partial}{\partial \phi_i}$$

and

$$V(p) = \sum_{i=1}^{n} V_i^{\psi}(p) \frac{\partial}{\partial \psi_i}$$

For all $p \in U \cap V$. Work out the relation between the coefficients $V_i^{\phi}(p)$ and $V_i^{\psi}(p)$.

Suggested Exercise 2. (0 points)
Is every bijective differentiable map a diffeomorphism? (Prove this or give a counterexample.)

Suggested Exercise 3. (0 points)
Let $\mathbb{CP}^n = \mathbb{C}^{n+1} \setminus \{0\}/\sim$ where $x \sim y$ iff $y = \lambda x$ for $\lambda \in \mathbb{C} \setminus \{0\}$. Consider the space

$$\gamma_n = \{([x], y) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \,|\, y = \lambda x \qquad \text{for some} \qquad \lambda \in \mathbb{C}\}$$

and the projection $\pi : \gamma_n \to \mathbb{CP}^n$ given by $\pi([v], y) = [v]$. This is called the tautological line bundle. Show that the tautological line bundle is a complex line bundle.

Hand in: Monday 8nd May
in the pigeonhole
third floor MI